Name of College - S. S. College, Jehanabad
Deptt - Mathematics

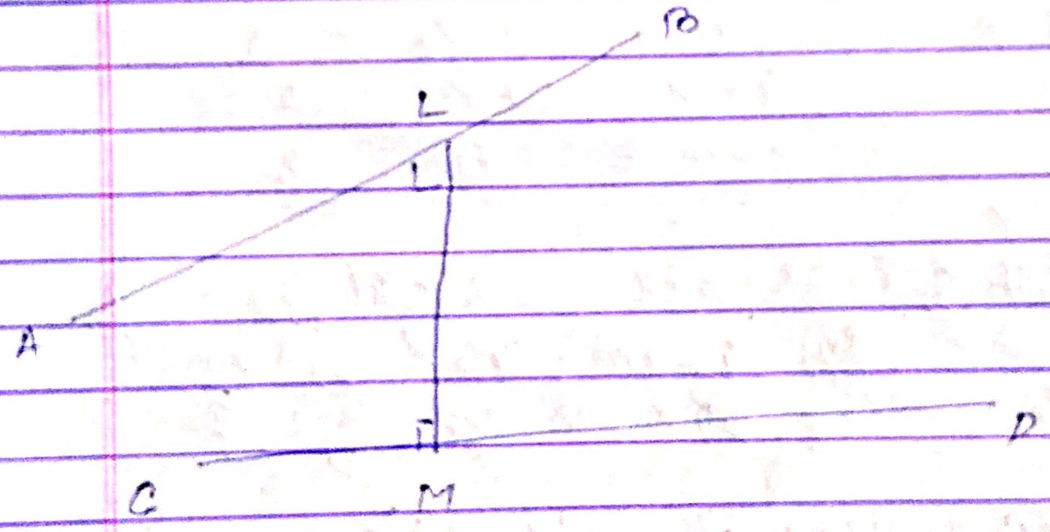
Topic - Problem on the Shortest distance.
(Analytical Geometry of 3-D)

Class - B.Sc I (Sub + Hons)
Time - 1:00 p.m to 1:45 p.m
Date - 18-08-2020

By - Dr. Shree Nath Sharma

Problem - Find the shortest distance between the lines

$$\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}, \quad \frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$$



Solution :→ Let AB and CD be the given st. lines.

Where the equation of AB

$$\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$$

and equation of CD is

$$\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$$

Say $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1} = r$ ——— (I)

$\Rightarrow x = 3r+3$ ∴ Any Point L on AB is
$y = -r+8$ $(3r+3, -r+8, r+3)$
$z = r+3$

The Equations to St. line CD

$\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4} = k$ ——— (II)

$\Rightarrow x = -3k-3$
$y = 2k-7$ Thus Any point M on CD is
$z = 4k+6$ $(-3k-3, 2k-7, 4k+6)$

∴ Direction ratios of LM

$(3r+3+3k+3, \; -r+8-2k+7, \; r+3-4k-6)$

$\Rightarrow (3r+3k+6, \; 15-r-2k, \; r-4k-3)$

(Applying $x_2-x_1, y_2-y_1, z_2-z_1$ direction ratio joining the line (x_1,y_1,z_1) and (x_2,y_2,z_2))

Since LM is perpendicular to both the lines AB and CD

Direction ratio of AB $(3, -1, 1)$
Direction ratio of CD $(-3, 2, 4)$

∴ $3(3r+3k+6) - (-r-2k+15) + 1\cdot(r-4k-3) = 0$

{Since $a_1a_2 + b_1b_2 + c_1c_2 = 0$ where two lines are mutual perpendicular having direction ratios (a_1, b_1, c_1) and (a_2, b_2, c_2)

$\Rightarrow 9r+9k+18+r+2k-15+r-4k-3=0$

$11r+7k = 0$ ——— (III)

Also $-3(3r+3k+6) + 2(15-r-2k) + 4(r-4k-3) = 0$

$-9r-9k-18+30-2r-4k+4r-16k-12=0$

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$-7r - 29k = 0$ ——— (4)

$\Rightarrow 7r = -29k$

$\Rightarrow r = -\frac{29}{7}k$

put $r = -\frac{29k}{7}$ in $11r + 7k = 0$

$\Rightarrow -\frac{11 \times 29}{7}k + 7k = 0$

$\Rightarrow (-319 + 49)k = 0$

$\Rightarrow -270k = 0$

$k = 0$

∴ putting $k = 0$

in $r = -\frac{29}{7}k$

$= -\frac{29}{7} \times 0 = 0$

Thus Co-ordinate of L is $(3, 8, 3)$

Thus co-ordinate of M is $(-3, -7, 6)$

$\left(PQ = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2} \right)$

$\therefore LM = \sqrt{6^2 + 15^2 + 3^2}$

$= \sqrt{270} = 3\sqrt{30}$

Problem :→ Find the length and equations of the shortest distance between the lines

$3x - 9y + 5z = 0 = x + y - z$

and $6x + 8y + 3z - 13 = 0 = x + 2y + z - 3$

Here First line is

$3x - 9y + 5z = 0$

$x + y - z = 0$

$\Rightarrow \frac{x}{9-5} = \frac{y}{5+3} = \frac{z}{3+9}$

$\Rightarrow \frac{x}{4} = \frac{y}{8} = \frac{z}{12}$

$\Rightarrow \frac{x}{1} = \frac{y}{2} = \frac{z}{3}$

this for equation of AB is

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3} = r \quad \text{——} (1)$$

$\Rightarrow x = r, \quad y = 2r, \quad z = 3r$

Let P be any point on AB then co-ordinates of P is $(r, 2r, 3r)$

Now we shall reduce the equation of the 2nd line say CD

$$6x + 8y + 3z - 13 = 0$$
$$x + 2y + z - 3 = 0 \quad \text{in symmetrical form}$$

Cut this line by a plane $z = 0$

then $6x + 8y - 13 = 0$

$x + 2y - 3 = 0$

using cross multiplication

$$\frac{x}{-24 + 26} = \frac{y}{-13 + 18} = \frac{1}{12 - 8}$$

$$\Rightarrow \frac{x}{2} = \frac{y}{5} = \frac{1}{4}$$

$$\therefore x = \frac{1}{2}, \quad y = \frac{5}{4}, \quad z = 0$$

let l, m, n be the direction ratios of the second line

then $6l + 8m + 3n = 0$

$l + 2m + n = 0$

By cross multiplication

$$\frac{l}{8 - 6} = \frac{m}{3 - 6} = \frac{n}{12 - 8}$$

$$\Rightarrow \frac{l}{2} = \frac{m}{-3} = \frac{n}{4}$$

this equation of 2nd line CD is in symmetrical form

$$\frac{x - 1/2}{2} = \frac{y - 5/4}{-3} = \frac{z - 0}{4} \quad \text{——} (11)$$

The Equation of plane through (1) and parallel to (2) is

$$\begin{vmatrix} x & y & z \\ 1 & 2 & 3 \\ 2 & -3 & 4 \end{vmatrix} = 0$$

$$x\begin{vmatrix} 2 & 3 \\ -3 & 4 \end{vmatrix} - y\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} + z\begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix} = 0$$

$\Rightarrow (8+9)x - (4-6)y + (-3-4)z = 0$

$\Rightarrow 17x + 2y - 7z = 0$ ——— (111)

$\therefore$ the required shortest distance

= perpendicular distance of the point $(\frac{1}{2}, 5/4, 0)$ from (3)

$$= \frac{17 \times \frac{1}{2} + 2 \times \frac{5}{4} - 7 \cdot 0}{\sqrt{(17)^2 + (5/4)^2 + (-7)^2}} = \frac{11}{\sqrt{342}}$$

(on calculation)

The direction cosines of shortest distance which is perpendicular to (3) are proportional to (17, 2, -7)

The Equation of the plane through (1) and the shortest distance is

$$\begin{vmatrix} x & y & z \\ 1 & 2 & 3 \\ 17 & 2 & -7 \end{vmatrix} = 0$$

On calculation, we get $10x - 29y + 16z = 0$

The Equation of the plane through (71) and the shortest distance is

$$\begin{vmatrix} x-\frac{1}{2} & y-\frac{5}{4} & z \\ 2 & -3 & 4 \\ 17 & 2 & -7 \end{vmatrix} = 0$$

On calculation we get—

$$13x + 82y + 55z - 109 = 0$$

Hence the Equations to the line of shortest distance are

$$10x - 29y + 16z = 6$$
$$13x + 82y + 55z - 109 = 0$$

Problem – Show that the equation to the plane containing the line

$\frac{y}{b} + \frac{z}{c} = 1, \quad x = 0$ and parallel to

the line $\frac{x}{a} - \frac{z}{c} = 1, \quad y = 0$ is

$\frac{x}{a} - \frac{y}{b} - \frac{z}{c} + 1 = 0$, and if 2d is the

shortest distance, prove that

$$\frac{1}{d^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$$

Solution :→ The Equation of any plane through the first line is

$y/b + \frac{z}{c} - 1 + \lambda x = 0$ ——— (I)

The plane (I) will be parallel to the second line if it will be parallel to plane through the second line

The equation of any plane through the second line is

$$\frac{x}{a} - \frac{z}{c} - 1 + Ky = 0 \text{ ——— (II)}$$

The planes (I) and (II) are parallel if the normals to them are parallel which is possible only when.

$$\frac{\lambda}{1/a} = \frac{1/b}{K} = \frac{\frac{1}{c}}{-1/c}$$

$$\frac{\lambda}{1/a} = \frac{1/b}{K} = -1$$

$$\lambda = -\frac{1}{a} \quad K = \frac{1}{b}$$

Put $\lambda = -1/a$ in (I)

$$\frac{y}{b} + \frac{z}{c} - 1 - \frac{x}{a} = 0$$

$$\frac{x}{a} - y/b - z/c + 1 = 0 \text{ ——— (III)}$$

Which is the equation of the required plane.

The shortest distance between them is the distance of any point on (I) from the plane (III)

$\therefore$ $2d$ = Perpendicular from $(a,0,0)$

$$= \frac{2}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}$$

$$\Rightarrow d = \frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}$$

$$d^2 = \frac{1}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}$$

$$\therefore \frac{1}{d^2} = \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$$